REAL FUNCTIONS: local analysis

Two weeks

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Module

This chapter is part of the APEF module: Analysis, Polynomials, Differential Equations, Functions. It is in the B3.

The **prerequisites** are:

- Finding the derivatives of usual functions and of their compositions.
- Comparing two sequences, using Landau comparisons (SR module, B2).

Motivation

When a program does a numerical computation, it will often call a real function, that is, a function taking a real number as argument and returning a real number. Such a function can be the result of a complex code. However, it is sometimes easier to understand this function if you can get an approximation.

Indeed, the real functions have a particular property that differs from other functions studied in the seminar: their input and output sets are $\mathbb{R}$ or, at least, intervals from $\mathbb{R}$. Those are sets where we can define things like "x approaches x_0" (where x_0 is a given value in $\mathbb{R}$), or "$f(x)$ tends to a value y_0 as x approaches x_0". We can hence study the behavior of a function as x approaches a given value x_0.

You have seen a similar situation during the S1, while studying numerical sequences. A sequence is a function from $\mathbb{N}$ to $\mathbb{R}$. The input n cannot approach a value $n_0 \in \mathbb{N}$, but it can approach $+\infty$. Furthermore, since its output set is $\mathbb{R}$, the sequence (u_n) may have a limit as n approaches $+\infty$. Thus, you have studied some sequence behaviors: does the sequence converge? If not, does it admit an infinite limit? You have also seen how we can compare two sequences, using the Landau comparisons: $u_n = o(v_n)$, $O(v_n)$ or $u_n \sim v_n$.

Globally, studying the behavior of a function f as x approaches a value $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, or of a sequence (u_n) as n approaches $+\infty$, leads to find a simple approximation. The first example we can imagine is when there exists a limit $\ell \in \mathbb{R}$: an approximation of $f(x)$ as x approaches x_0 can be this constant value ℓ. But this not always possible: it may happen that there is no limit. Furthermore, when the limit exists, this approximation may be too poor: an approximation implies an approximation error, and we may need a smaller error, that is, a more precise approximation which would not be a constant.

The approximations that we will use are the Taylor expansions. The purpose of this chapter is to be familiar with them. The word "approximation" has to be interpreted in its global sense: it includes ***both*** the simple approximating formula (most of the time, a polynomial) ***and*** the error term. You will see how to get Taylor expansions and you will train to handle them.

Learning outcomes

The skills that you're expected to get in this chapter are the following:

- Proving and handling comparison relations between two function, as x approaches a given value or ∞ (Laudau comparisons).
- Finding some Taylor expansions, using the basic Taylor expansions, for example to find some limits and to solve undermined forms.

More precisely, you should be able to:

- Compare two functions as x approaches a given value or ∞.

- Add, multiply and divide comparison relations between two functions.
- Enumerate **safely** the basic Taylor expansions as x approaches 0.
- Compute the Taylor expansion of a function using basic Taylor expansions.
- Add, multiply and compose Taylor expansions.
- Anticipate at which order to expand a function, depending on the question you have to answer.
- Never forget the approximation error, and be able to quantify it

Take care: forgetting a remainder (small o) in a Taylor expansion will be strongly penalized at exams, as well as using the symbol = instead of $\sim$.

1 Landau comparisons

1.1 Summary

You have studied these comparisons when working with numerical sequences: you trained to compare two sequences (u_n) and (v_n) as n approaches $+\infty$. Remind that:

1. $u_n = o(v_n)$ if there exists a sequence (ε_n) converging to 0 such that $u_n = v_n\varepsilon_n$.
   If the sequence (v_n) does not vanish, then $(\varepsilon_n) = \left(\frac{u_n}{v_n}\right)$ and: $\quad u_n = o(v_n) \iff \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 0$

2. $u_n = O(v_n)$ if there exists a bounded sequence (b_n) such that $u_n = v_n b_n$.
   If the sequence (v_n) does not vanish, then $(b_n) = \left(\frac{u_n}{v_n}\right)$ and: $\quad u_n = O(v_n) \iff \left(\frac{u_n}{v_n}\right)$ is bounded.

3. $u_n \sim v_n$ if there exists a sequence (ε_n) converging to 0 such that $u_n = v_n(1+\varepsilon_n)$.
   If the sequence (v_n) does not vanish, then $(\varepsilon_n) = \left(\frac{u_n}{v_n} - 1\right)$ and: $\quad u_n \sim v_n \iff \frac{u_n}{v_n} \xrightarrow[n\to+\infty]{} 1$

These definitions are naturally extended to the comparison of two functions f and g as the variable x approaches a given $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition that you have to know rigorously:
We say that, as x approaches x_0:

1. $f(x) = o\big(g(x)\big)$ if there exists a function ε tending to 0 such that $f(x) = g(x)\varepsilon(x)$.
   If the function g does not vanish, then $\varepsilon = \frac{f}{g}$ and:

$$f(x) = o\big(g(x)\big) \iff \frac{f(x)}{g(x)} \xrightarrow[x\to x_0]{} 0$$

2. $f(x) = O\big(g(x)\big)$ if there exists a bounded function b such that $f(x) = g(x)b(x)$.
   If the function g does not vanish, then $b = \frac{f}{g}$ and:

$$f(x) = O\big(g(x)\big) \iff \frac{f}{g} \text{ is bounded}$$

3. $f(x) \sim g(x)$ if there exists a function ε tending to 0 such that $f(x) = g(x)\big(1+\varepsilon(x)\big)$.
   If the function g does not vanish, then $\varepsilon = \frac{f}{g} - 1$ and:

$$f(x) \sim g(x) \iff \frac{f(x)}{g(x)} \xrightarrow[x\to x_0]{} 1$$

In this section, we see some examples and we study the behavior of these comparisons regarding the usual operations with real functions: addition/substraction, multiplication/division, composition.

1.2 Exercises

Exercise 1.1
Consider three functions f, g and h such that, as x approaches $x_0 = 0$,

$$f(x) = o(0), \qquad g(x) = O(0) \qquad \text{and} \qquad h(x) \sim 0$$

What can you say about these functions?

Exercise 1.2
Let f be a function defined on $\mathbb{R}$, $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $\ell \in \mathbb{R}$. We focus on the properties of f as x approaches x_0.

Is the proposition (P) below true of false?

$$(P): \qquad f(x) \sim \ell \iff f(x) \xrightarrow[x \to x_0]{} \ell$$

Exercise 1.3

1. Let $(\alpha, \beta) \in \mathbb{R}^2$. Find a condition on α and β to have:
   (a) $x^\alpha = o\left(x^\beta\right)$ as x approaches $+\infty$?
   (b) $x^\alpha = o\left(x^\beta\right)$ as x approaches 0?
2. Let f be the function defined on $\mathbb{R}$ by: $f(x) = 2x - x^2 - 4x^3$.
   (a) Find a simple equivalent of $f(x)$ in $+\infty$.
   (b) Find a simple equivalent of $f(x)$ in 0.

Exercise 1.4
Compare, in 0 and in $+\infty$:

1. $\ln(x)$ and x
2. e^x and x
3. $2x^3 - x^2 - x$ and $x^2 + 1$

Exercise 1.5
Find a simple equivalent of:

1. $2x^8 - 3x^7 + 10$ as x approaches 0, then $+\infty$
2. $2x^8 - 3x^7$ as x approaches 0, then $+\infty$
3. $\sin(x)$ as x approaches 0. Hint: what is $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x}$?
4. e^x as x approaches 0.
5. $e^x - 1$ as x approaches 0. Hint: what is $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$?

Exercise 1.6

Consider two functions f and g such that, in 0,

$$f(x) = o(x) \qquad \text{and} \qquad g(x) = o\left(x^2\right)$$

1. Write the possible comparisons, if any, between:
   (a) $2f(x)$ and x
   (b) $f(x)$ and $2x$
   (c) $f(x)$ and $g(x)$
2. For each of the following functions, is there an $\alpha \in \mathbb{R}$ such that the function is $o(x^\alpha)$? If there is, give its value.
   (a) $f + g$
   (b) $f - g$
   (c) fg
   (d) $\dfrac{f}{g}$ (assuming that g does not vanish)
   (e) xf
   (f) $\dfrac{g}{x}$

Exercise 1.7

Consider four functions f, g, h and k defined on $\mathbb{R}$ such that, as x approaches a given $x_0 \in \mathbb{R} \cup \{-\infty, \infty\}$,

$$\begin{cases} f & \sim & g \\ h & \sim & k \end{cases}$$

Which of the following properties can we deduce? Justify your answers.

1. $f + h \sim g + k$
2. $fh \sim gk$
3. $\frac{f}{h} \sim \frac{g}{k}$ (assuming that the functions h and k don't vanish)
4. $|f| \sim |g|$
5. for all $\alpha \in \mathbb{R}$, $f^\alpha \sim g^\alpha$
6. $e^f \sim e^g$
7. $\ln(f) \sim \ln(g)$ (assuming that the functions f and g are strictly positive)

Exercise 1.8

Find simple equivalents of the following functions:

1. $\dfrac{2x^3 - x + 6}{1 - x}$ in 0, then in $+\infty$
2. $\sin(x) \times (e^x - 1) \times (3x^2 + 2x)$ in 0
3. $\sin(x) + x^2$ in 0, then in $+\infty$

Exercise 1.9

1. Consider a function f such that, as x approaches 0, $f(x) = 2x - \dfrac{x^3}{2} + o\left(x^3\right)$.
   Can we deduce simple equivalents of the following functions?
   (a) $f(x)$
   (b) $f(x) - 2x$
   (c) $f(x) - 2x + \dfrac{x^3}{2}$
2. Suppose now that $f(x) = 2x - \dfrac{x^3}{2} + \dfrac{x^4}{8} + o\left(x^4\right)$.
   (a) Does this hypothesis contradict the hypothesis of part 1?
   (b) Answer the questions of part 1, using this new hypothesis.

2 Taylor expansions

2.1 Summary

When a real function f satisfies some regularity conditions, it is possible to get a polynomial approximation of f as x approaches a given value $x_0 \in \mathbb{R}$. Taylor-Young theorem states that, if f is of class $\mathcal{C}^n$ on an interval I of $\mathbb{R}$ and if x_0 is a given value in I, then as x approaches x_0,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o\left((x - x_0)^n\right)$$

This formula leads to nested approximations, each one being more precise than the previous one:

$$\begin{aligned} f(x) &= f(x_0) + o(1) \\ &= f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + o\left((x - x_0)^2\right) \\ &\vdots \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o\left((x - x_0)^n\right) \end{aligned}$$

In practical, Taylor-Young theorem is only used to get the Taylor expansions of the basic functions. A more complex function is usually a combination of the latter ones. We get its Taylor expansion by doing the same combination with the basic Taylor expansions.

This section focuses on these operations: addition, multiplication and composition of Taylor expansions. The final results can be used, for example, to solve some difficult limits.

2.2 Exercises

Exercise 2.10
Recall the Taylor expansions in 0, up to the order 4, of the following functions:

1. $f(x) = e^x$.
2. $g(x) = \ln(1 + x)$.
3. $h(x) = (1 + x)^\alpha$ where $\alpha \in \mathbb{R}^*$.

4. $i(x) = \sin(x)$.

5. $j(x) = \cos(x)$.

Exercise 2.11
Find the Taylor expansions in 0 of the following functions:

1. $f(x) = \dfrac{1}{1+x} - e^x$ at the order 3.

2. $g(x) = e^x \cos(x)$ at the order 4.

3. $h(x) = \sin(3x)\ln(1-x)$ at the order 3.

4. $i(x) = e^{\sin(x)}$ at the order 4.

5. $j(x) = e^{\cos(x)}$ at the order 3.

6. $k(x) = \sqrt{1+2x}$ at the order 4.

7. $l(x) = \ln(2+x)$ at the order 3.

8. $m(x) = \ln(1+e^x)$ at the order 2.

9. $n(x) = (1+x)^{\frac{1}{x}}$ at the order 2.

10. $o(x) = \Big(\cos(x)\Big)^{\sin(x)}$ at the order 4.

★ **Exercise 2.12**
Same question for:

$f(x) = \dfrac{1}{\sqrt{2+x}}$ at the order 3, $g(x) = \ln(1+\cos(x)+\sin(x))$ at the order 3, $h(x) = \tan(x)$ at the order 5.

Exercise 2.13
Study the existence of the following limits. When they exist, find their values.

1. $\lim\limits_{x\to 0} \dfrac{e^x - \cos(x) - x}{x - \ln(1+x)}$.

2. $\lim\limits_{x\to 0} \dfrac{e^x - \cos(x)}{x^2}$.

3. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\ln(1+x)}$.

4. $\lim\limits_{x\to 0} \dfrac{\sin^2(x) - x\ln(1+x)}{e^x + \cos(x) - \sin(x) - 2}$.

5. $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x}$.

6. $\lim\limits_{x\to +\infty} \left(1 + \dfrac{a}{x}\right)^x$ where $a \in \mathbb{R}$.

7. $\lim\limits_{x\to +\infty} \left(\cos\left(\dfrac{1}{x}\right)\right)^{x^2}$.

8. $\lim\limits_{x\to +\infty} x^3 \sin\left(\dfrac{1}{x}\right) - x^2$.

9. $\lim\limits_{n\to +\infty} \left(\dfrac{n}{n-42}\right)^n$.